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SYMMETRIES IN MUSIC TEACHING

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Abstract—The study presents some simple possibilities of symmetrical structures of the same principle either in small or large pieces of music.

In part–whole relationships of musical manifestations there are symmetries in dynamics, rhythm, melody, harmony, tonality and form. It is one of the crucial tasks of music pedagogy to recognize and practice them in a creative way.

SYMMETRIES IN MUSIC TEACHING

The present paper touches upon some problems of symmetries in music, employing experiences obtained in teaching.

The most frequent types of symmetry in music:

(a) bilateral (essentially, it is a reflection across a vertical or horizontal axis);
(b) translational (it can involve shifting, reversing, or translocation which is motion implying repetition of a certain fundamental unit). Practically, it is the symmetry of recurrence resulting from shifting, the consequence of which is that the fundamental unit overlaps itself time and time again.
(c) rotational (it means turning around across an axis with the end being the starting position).

Before pointing out symmetries in actual compositions, let us outline the possibilities of symmetrical construction of musical elements.

By musical element we mean tone as the base of every kind of sound. Tones are determined by their intensity, duration, pitch and timbre. All of them may constitute different symmetries as well.

The parameters mentioned above interrelated, and together they characterize tones. However, here we are discussing the possible symmetries formed by intensity, duration and pitch separately.

SYMMETRY OF INTENSITY

Intensity of tone is called dynamics, which has a wide range from the lowest to the loudest. It is influenced considerably by spatial situations, consequently a gradual dynamical increase makes us feel as if we are coming nearer, while a decrease makes us feel as if we are moving away.

(a) In Fig. 1 there is a gradual strengthening and softening, respectively, at continuous sounding of the same pitch. This process represents a reflection across a vertical axis, which, practically, results in bilateral symmetry.

(b) Bilateral symmetry is obtained by uniformly interrupted repetition (staccato) of tone of the same pitch, however, with different dynamics (Fig. 2).

(c) Gradual strengthening of tones of the same pitch and repetition of the units result in translational symmetry (Fig. 3). In this case, the three tones of intensifying dynamics make the fundamental unit.

(d) Multiple repetition of action denoted under (a) leads partly to translation, and partly to rotation (Fig. 4). The same can be illustrated in another figure to make rotation unambiguous (Fig. 5).

SYMMETRY OF DURATION—SYMMETRY OF RHYTHM

Duration of tones are denoted by their values, by which we can distinguish tones sounding for longer or shorter times. Depending on values of tones different rhythms can be formed.

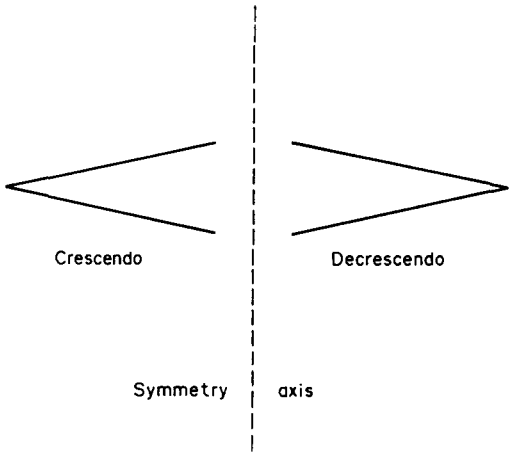


Fig. 1. Crescendo–decrescendo symmetry.

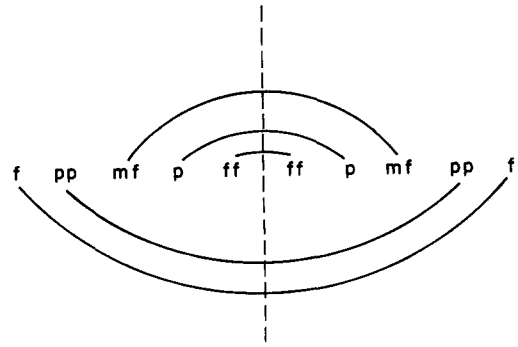


Fig. 2. Bilateral symmetry of different dynamics.

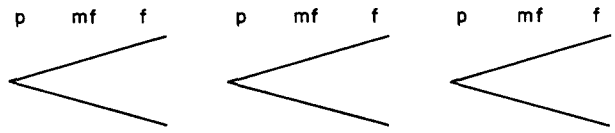


Fig. 3. Translational symmetry of gradual strengthening.

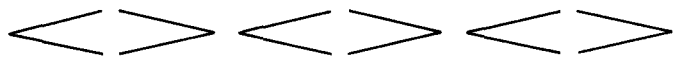


Fig. 4. Rotation of crescendo–decrescendo.

Symmetry can refer to both duration and rhythm.

(a) Tones of the same pitch and intensity with regular intervals can result in translation, where one tone makes up the fundamental unit (Fig. 6).

(b) Gradual acceleration and deceleration of tones of the same pitch and intensity creates bilateral symmetry. Spatial effect is unambiguous in this case (Fig. 7).

(c) Bilateral reflection of different rhythms (Fig. 8).

The example in Fig. 8 is taken from Minuet in C major by Mozart (namely its first four bars, which we will return to later). Mozart's minuet makes it obvious that, for example, ostinato (permanently repeated) rhythm means a translational symmetry.

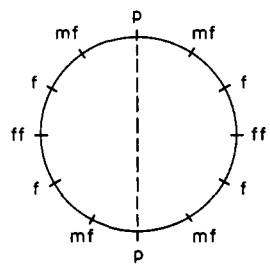


Fig. 5. Rotation of strengthening and softening illustrated by a ring.

Fig. 6. Translation of tones of same duration.

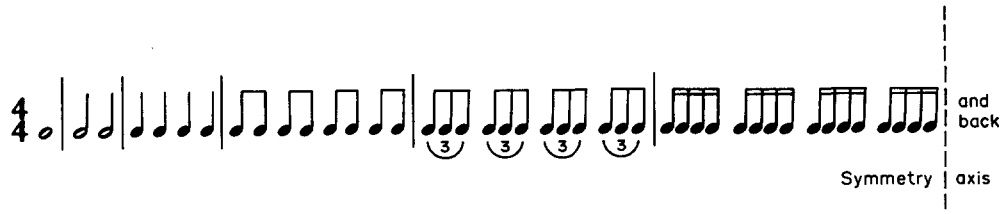


Fig. 7. Gradual acceleration and deceleration as bilateral symmetry.

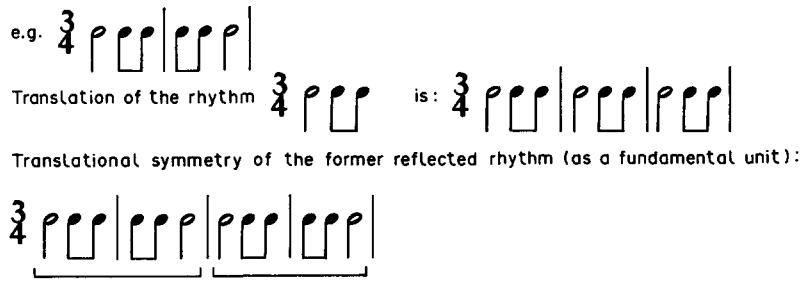


Fig. 8. Examples of bilateral and translational symmetries in rhythm.

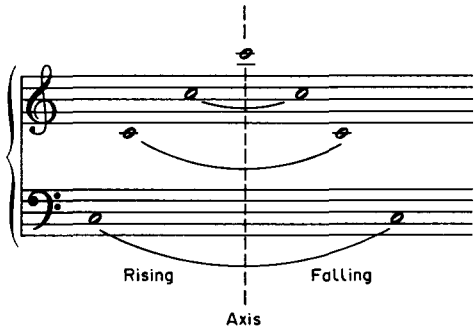


Fig. 9. Different pitches of the same tones as bilateral symmetry across a vertical axis.

PITCH SYMMETRY—MELODY SYMMETRY

Pitch is characteristic of tones, e.g. the order in pitch of a certain tone refers to octaves. Tones of different pitch make up melody. Pitch also denotes spatial relations, e.g. the terms high and low tone.

(a) Different pitches of the same tone as bilateral symmetry across a vertical axis are shown in Fig. 9. The reflection is formed by the retrograde-part and built on succession in time.

(b) Different pitches of the same tone as bilateral symmetry across a horizontal axis appear in Fig. 10. Here the point is spatial simultaneity of rising and falling; however, it can appear as a succession in time as well.

(c) Translation of different pitches of the same tone is illustrated in Fig. 11.

(d) Figure 12 shows rotation of different pitches of the same tone. In this case translation occurs as well. It can be observed that translation and rotation are often associated.

As an example of symmetries of intensity, rhythm and pitch we can use Prelude and Waltz in C from the series "*Games*" by György Kurtág (Fig. 13). Even this quite simple piece can prove that symmetric constructions manifest themselves in various ways, thus creating all of the musical form.

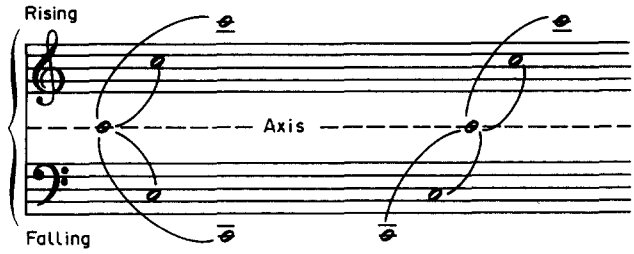


Fig. 10. Different pitches of the same tone as bilateral symmetry across a horizontal axis.

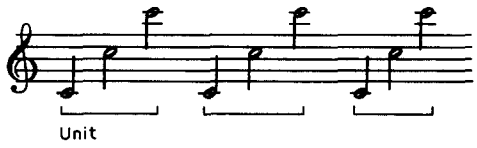


Fig. 11. Translation of different pitches of the same tone.

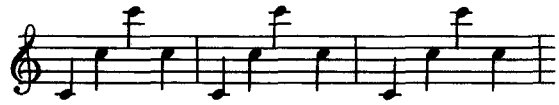


Fig. 12. Rotation of different pitches of the same tone.

In Prelude (Libero) the symmetry of pitch is rather concealed, and can be traced back to the basic situation (Fig. 14).

The symmetry of Waltz (Guisto) is built on the basic situation introduced at Prelude, i.e. C′ is the centre of symmetry, which is emphasized by the composer at the very end. Starting from C′ and corresponding to each other, the octaves of contrary direction succeed each other gradually. The rhythm of Waltz has translational symmetry (Fig. 15) while its dynamics have a bilateral one (Fig. 16).

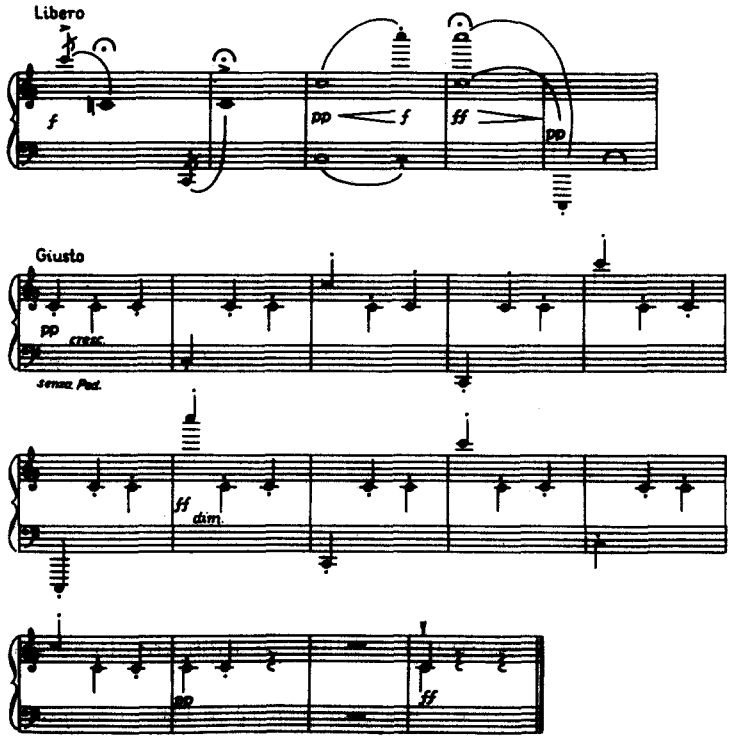


Fig. 13. Note of Prelude and Waltz in C by G. Kurtág.

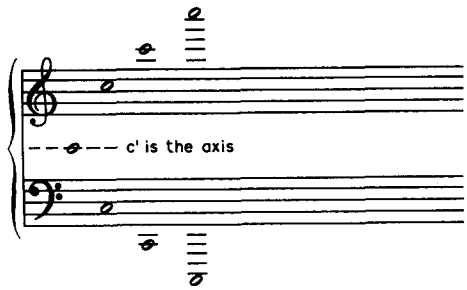


Fig. 14. Reflection of C tones across a horizontal axis.

Fig. 15. Crotchets—translational symmetry.

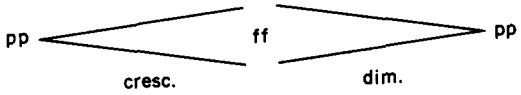


Fig. 16. Crescendo–diminution—bilateral symmetry.

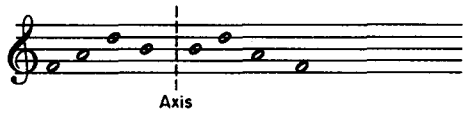


Fig. 17. Reflection of tones of different pitch across a vertical axis.

(e) Reflection of tones of different pitch across a vertical axis (Fig. 17). The tones reflected results in a so-called "retrograde-part" and succession in time can be seen. It can be illustrated by the sixth piece of "Six unisono melodies" from volume one of "*Microcosmos*" by Bartók (Fig. 18). The order of the tones is reflected in a retrograde-like way from C″ which is denoted as the axis. We can see clearly how symmetry and form are related within one work.

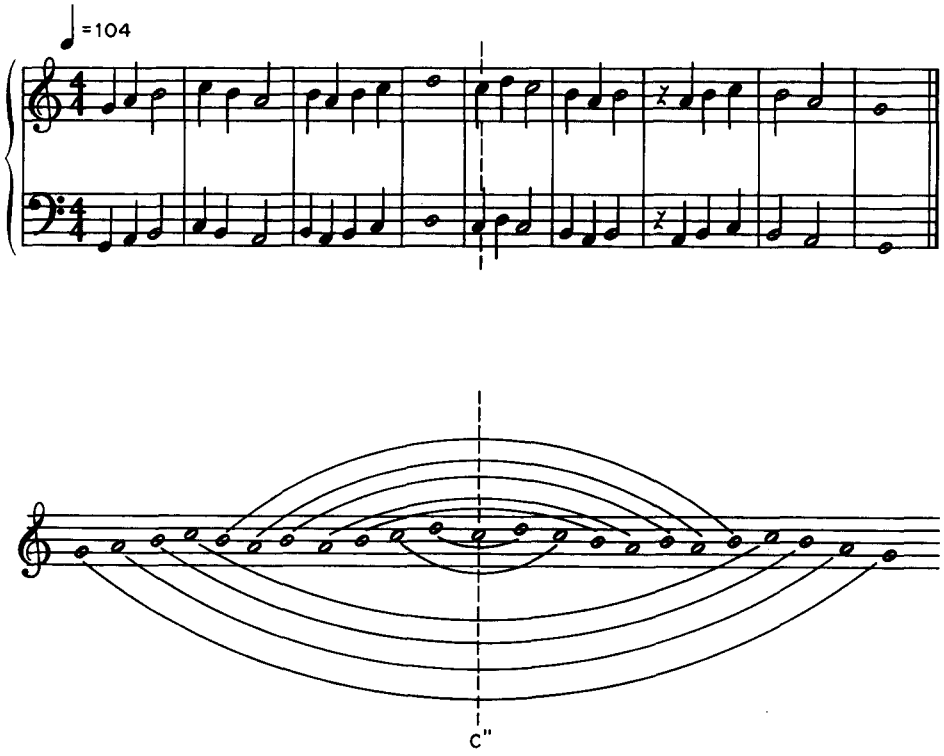


Fig. 18. Note of the sixth piece of "Six unisono melodies" from volume one of "*Microcosmos*" by Bartók and its figure of explanation.

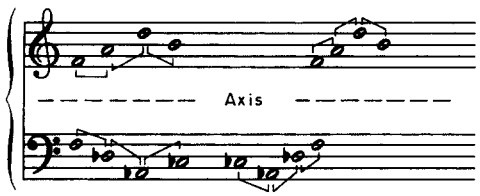


Fig. 19. Reflection of tones of different pitch across a horizontal axis.

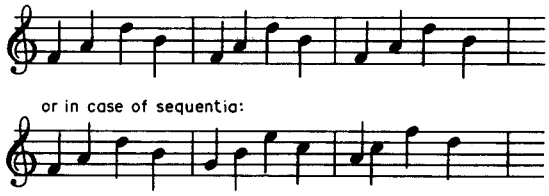


Fig. 20. Translation of tones of different pitch.

(f) Reflection of tones of different pitch across a horizontal axis (Fig. 19). The intervals of the low part coincide with those of the high part, however, they are of opposite direction. Both spatial simultaneity and succession in time are equally possible.

(g) Translation of tones of different pitch (Fig. 20). In both cases the question of how translational symmetry is formed related to rotation emerges. In musical structures we can refer to translational symmetry not only in the case of sequentia but also in the case of imitation, canon and fugue as well. Apart from these cases, we should mention here variational genres—where translation results from slight changes—and transposition—where the key of a piece of music is changed. However, no detailed discussion of this subject can be given at this time.

(h) Rotation of tones of different pitch (Fig. 21). Here too, rotation and translation intertwine.

Now let us have a look at some pieces of music where reflection across a horizontal axis can be found. The first is "*Leggiero*" by Lajos Papp. Its tonal system is a 1:5 scale model built on D. Here the distance between the tones is measured in halves (Fig. 22).

The next example is taken from "*Games*" by Kurtág ("Palm exercise"). Instead of single tones, here we find groups of tones. The place of the horizontal axis is obvious, as well as the central reflexion beginning at *pp*: the right hand goes from white keys to black keys, the left hand proceeds from black keys to white keys (♮0—white keys, #0—black keys). Here the rhythm becomes inverted as well (Figs 23 and 24). In the same way, if we have a look at the note of "Hommage á Paganini" (from "*Games*" by Kurtág) both the reflection across a horizontal axis and the correspondence of parts of contrary direction at the beginning and the end of the composition can be seen (Figs 25 and 26).

Examples of translational symmetry

First, Concerto (after Vivaldi) by Bach, a section from part three. Sequence is the characteristic structural principle of the work: the first two bars are repeated dominantly, a new sequence is started beginning from bar seven, and another one from bar 22 (Fig. 27).

In a section of part one of the same Bach Concerto symmetry is again created by sequences (Fig. 28). Another example can be taken from a passage of "Toccata in D major" by Bach (Figs 29 and 30).

In the following examples the sequence of repetitions and shiftings, namely rotation and translation, respectively, are intertwined:

Bach: Concerto (after Telemann), movement III, passage (Fig. 31)
Bach: Concerto (after unknown composer),
movement II, passage (Fig. 32)
Bach: Toccata in D major, passage (Fig. 33).

(i) Reflection of tones of different pitch across horizontal and vertical axes. This combined form of reflection is characteristic of dodecaphonic compositions. They are built on multiple

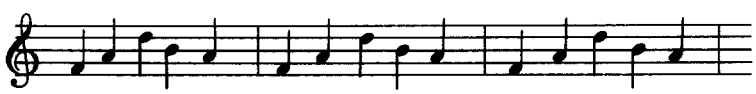


Fig. 21. Rotational tones of different pitch.

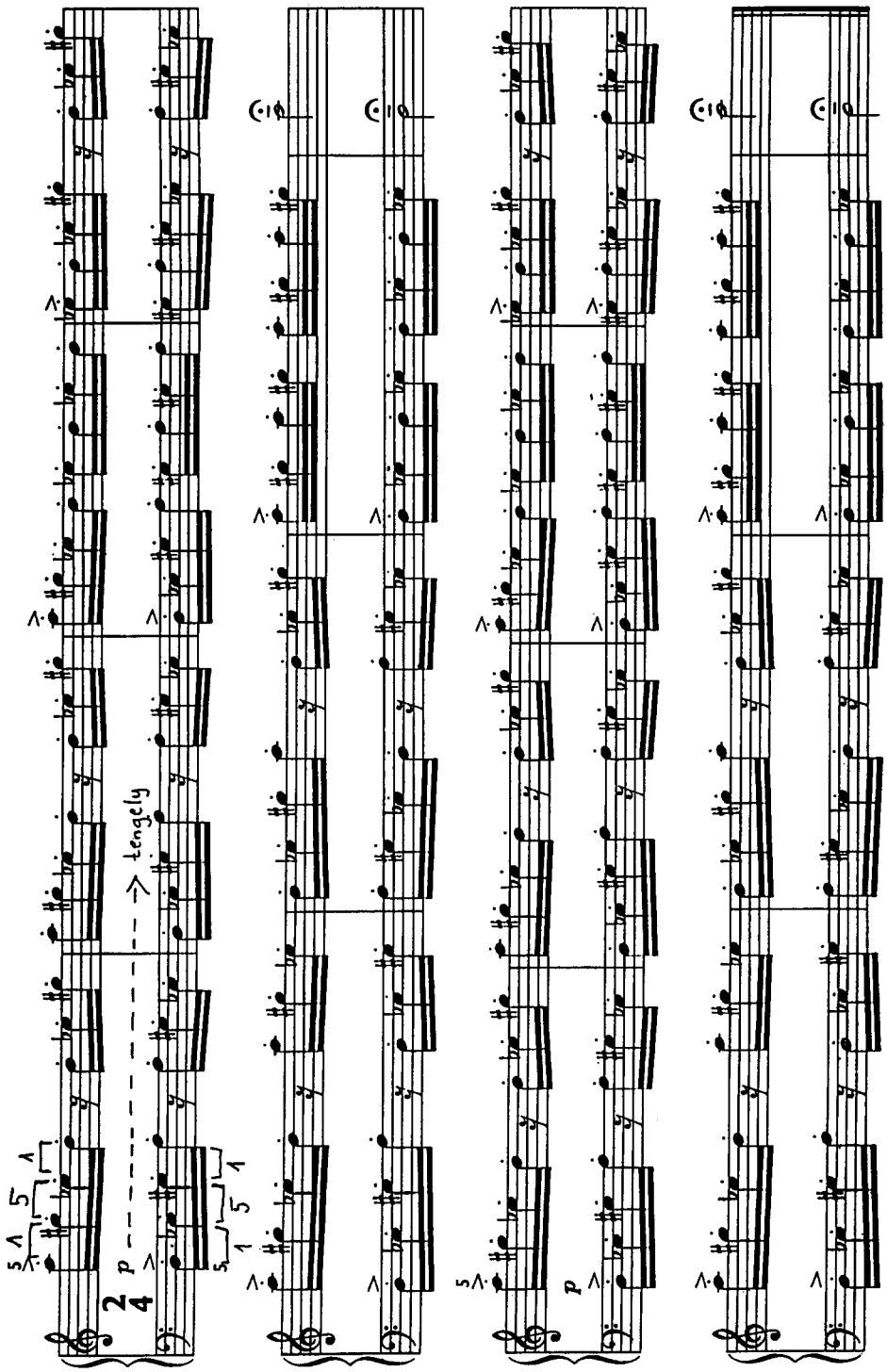


Fig. 22. Note of "*Leggiero*" by Lajos Papp.

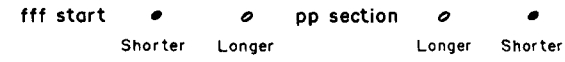


Fig. 23. Short–long, long–short rhythm-figure of explanation.

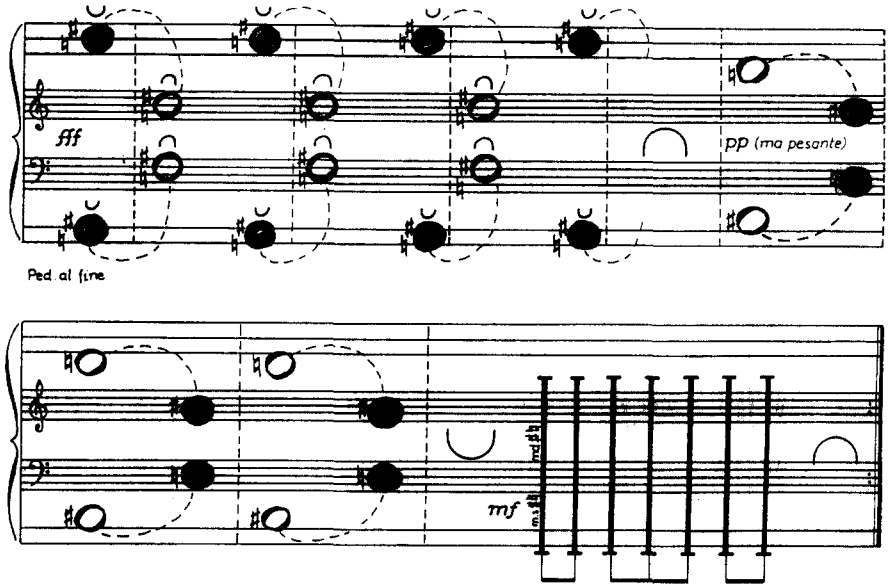


Fig. 24. Note of "Palm exercise" by G. Kurtág.

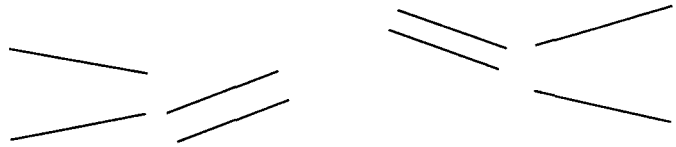


Fig. 25. Denotion of directions.

reflection of 12 scale Reihe, so we can see bilateral, central and translational symmetries as well. See Webern Variations for piano (Op. 27), movement I. Relations are denoted by numbers (Fig. 34).

Some pieces of the series for piano "Gradus" by Samuel Adler are also built on the Reihe technique. He created a table, to serve as a standby, in which he made each of the retrograde, inversion and retrograde inversion built on each tone of the basic line denoted as prime. In the first piece (8a) we can find the tones of the first prime line starting from F and those of the twelfth inversion-line built on C (Figs 35 and 36).

SYMMETRY OF HARMONY

Up to now, we have been discussing symmetries coming from tones as musical elements, which are related to intensity, duration, rhythm, pitch and melody.

However, one can often find a symmetric arrangement of harmonies in pieces of music. One of the most frequent pair in this respect is symmetric linkage of the tonic (T) and the dominant (D), which, in this case, means the I–V degrees and one of the inversions of V degree (Fig. 37).

Such an arrangement of harmonies can be observed in a great number of pieces of music. For example Sonata in C major (K 279), in G major (K 283), in D major (K 311) by Mozart; also in his *Fantasy* in D minor, or Minuet in C major; here we can enumerate *Ecossuise* in G major

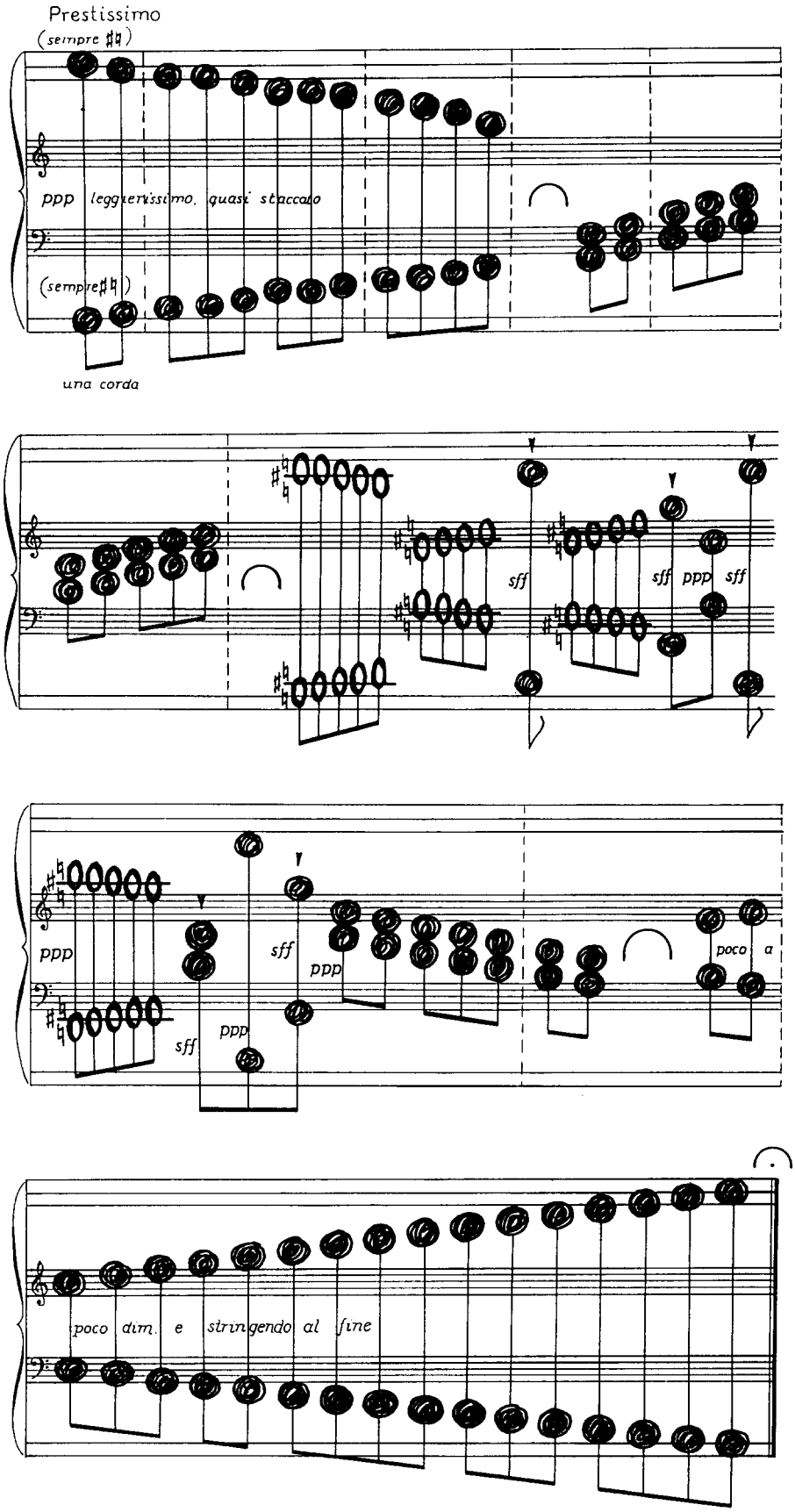


Fig. 26. Note of "Homage á Paganini" by Kurtág.

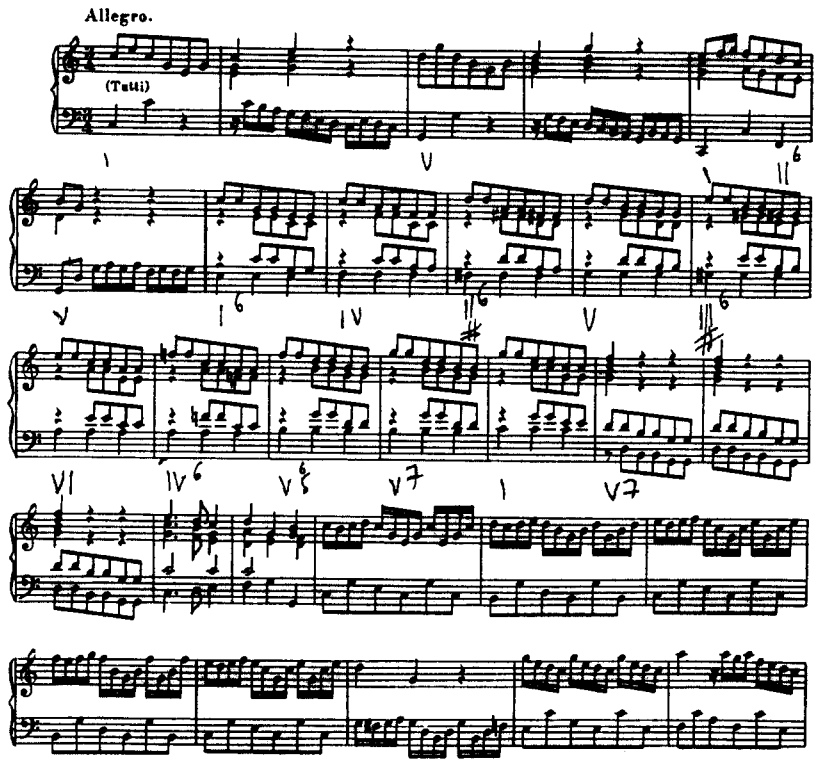


Fig. 27. Bach: Concerto, movement III, passage-note.

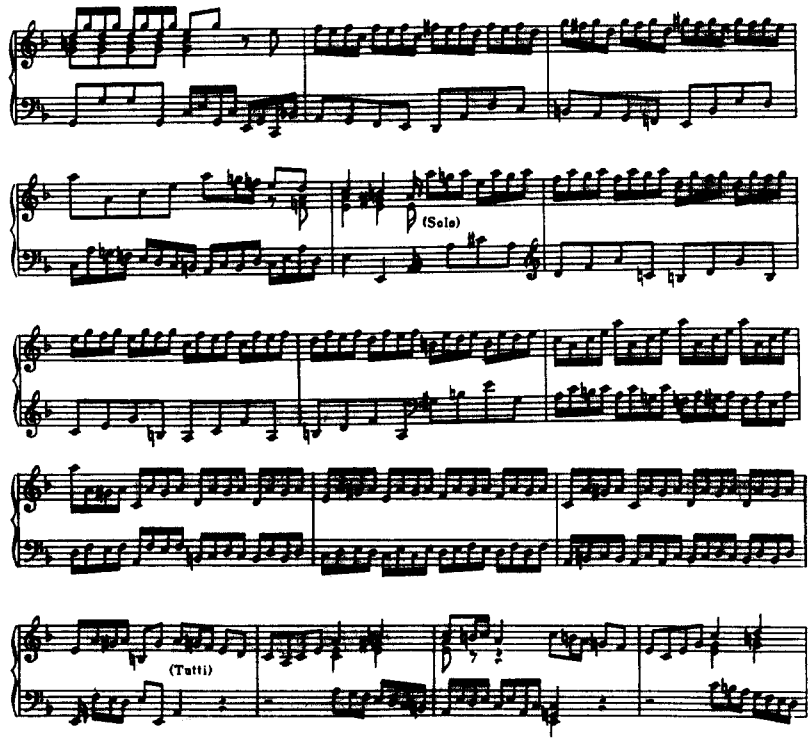


Fig. 28. Bach: Concerto, movement I, passage-note.

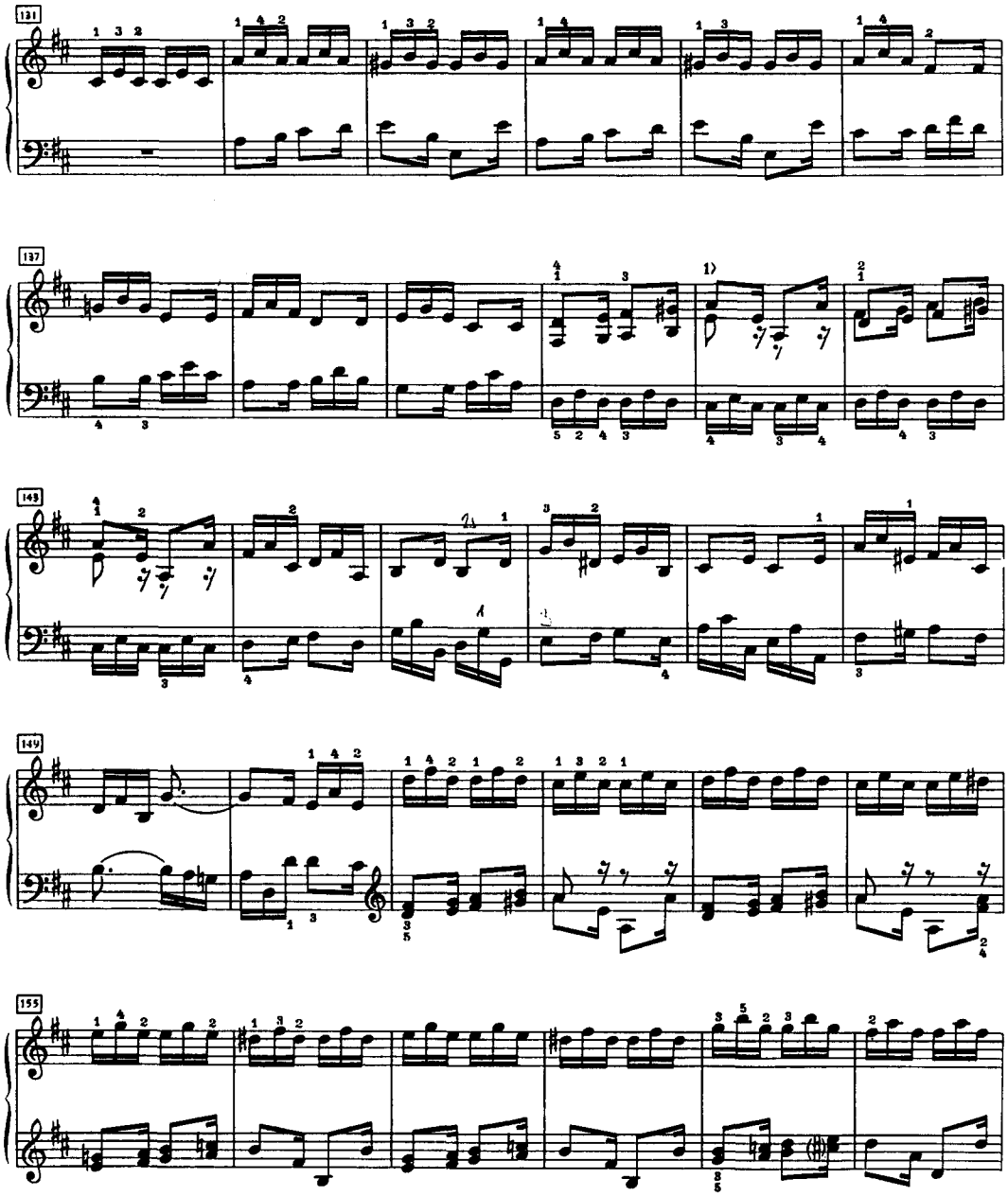
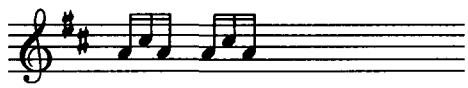


Fig. 29. Bach: Toccata, passage-note.

Translation of rhythmic units of the high part is transferred into the low part in bar 140, then in bar 151 it returns to the high part:



The rhythmic unit of the low part changes its place in the same way:

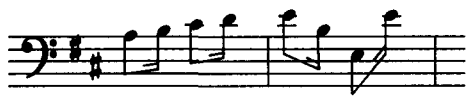


Fig. 30. Explanation with two short examples of note.

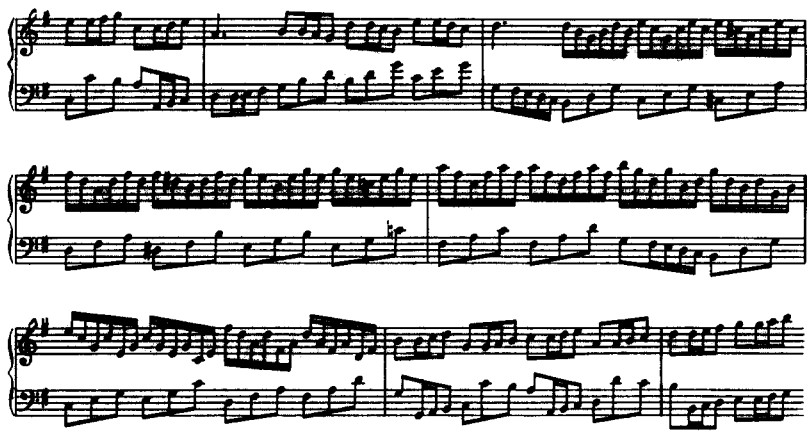


Fig. 31. Bach: Concerto, movement III, passage-note.

by Beethoven and *Ländler* in D major, *Ländler* in E flat major by Schubert. Of the above mentioned pieces, the last contains T–D functions only, and in its middle part one can find T D D T linkage (Fig. 38).

The harmony scheme of the piece is shown in Fig. 39. So, in this case, symmetry of harmony is in close relationship with symmetry of form.

SYMMETRY OF TONALITY

In tonal pieces of music, in which defined harmony functions (tonic, dominant, subdominant are connected, the starting mode often changes in the course of the piece, then it comes back. Schubert's latter *Ländler* starts from E flat major, then is modulated into B flat major, and again to the original key of E flat major (Fig. 40).

Modulation is possible not only into major key but into minor key as well; e.g. into the parallel minor. Schubert's *Letzte Walzer* in F major is modulated into the parallel minor: D minor. Its harmony scheme is shown in Fig. 41.

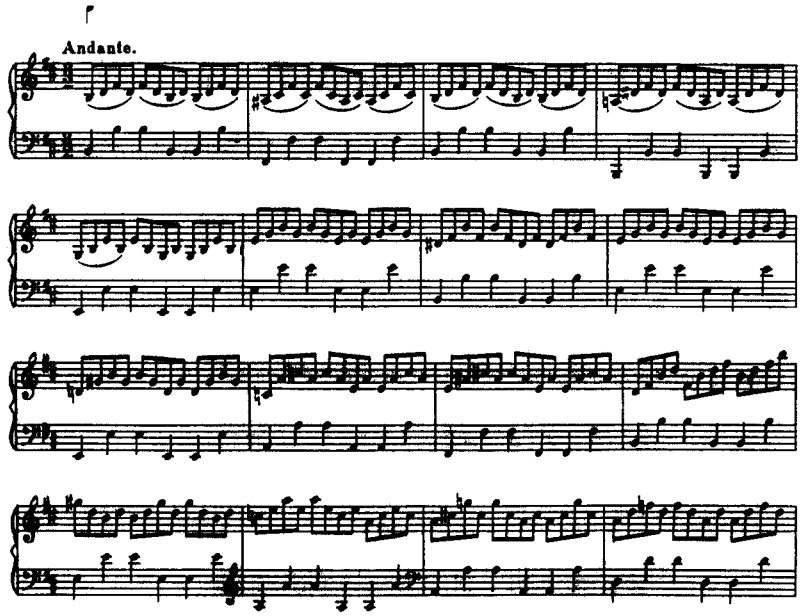


Fig. 32. Bach: Concerto, movement II, passage-note.

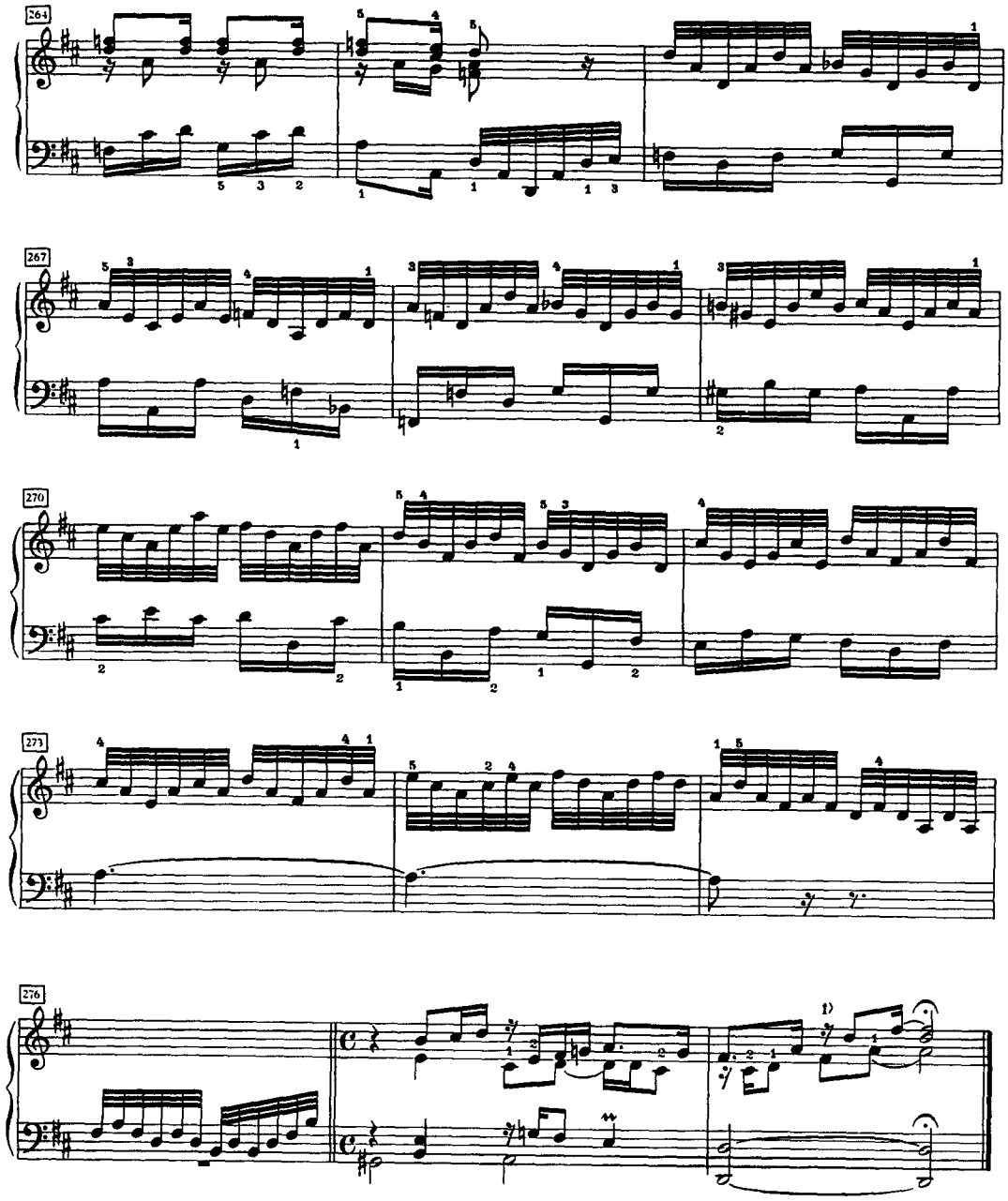


Fig. 33. Bach: Toccata, passage-note.

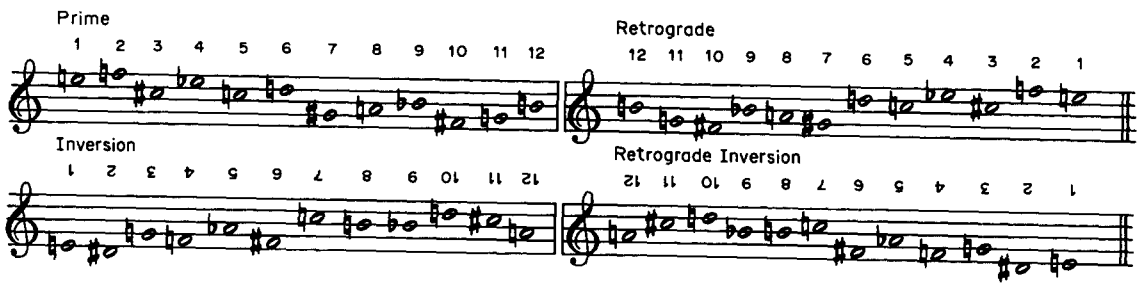


Fig. 34. Webern Variations for piano, Op. 27, movement I, first lines.

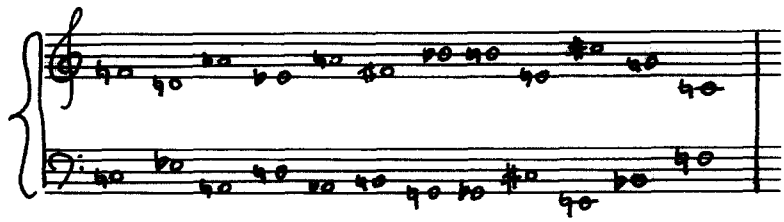


Fig. 35. Tones of the first prime line starting from F and tones of the inversion-line built on C

8

Prime →	1	2	3	4	5	6	7	8	9	10	11	12	← Retrograde
Inversion ↓ 1	F♮	D♮	A♭	E♭	A♮	F♯	B♭	B♮	E♮	C♯	G♮	C♮	1
2	A♭	F♮	B♮	F♯	C♮	A♮	D♭	D♮	G♮	E♮	B♭	E♭	2
3	D♮	B♮	F♮	C♮	F♯	D♯	G♮	G♯	C♯	B♭	E♮	A♮	3
4	G♮	E♮	B♭	F♮	B♮	G♯	C♮	C♯	F♯	D♯	A♮	D♮	4
5	D♭	B♭	E♮	B♮	F♮	D♮	G♭	G♮	C♮	A♮	E♭	A♭	5
6	E♮	D♭	G♮	D♮	A♭	F♮	A♮	B♭	E♭	C♮	F♯	B♮	6
7	C♮	A♮	E♭	B♭	E♮	D♭	F♮	G♭	B♮	A♭	D♮	G♮	7
8	B♮	A♭	D♮	A♮	E♭	C♮	E♮	F♮	B♭	G♮	D♭	G♭	8
9	F♯	D♯	A♮	E♮	B♭	G♮	B♮	C♮	F♮	D♮	G♯	C♯	9
10	A♮	F♯	C♮	G♮	C♯	B♭	D♮	D♯	G♯	F♮	B♮	E♮	10
11	E♭	C♮	G♭	D♭	G♮	E♮	A♭	A♮	D♮	B♮	F♮	B♭	11
12	B♭	G♮	D♭	A♭	D♮	B♮	E♭	E♮	A♮	G♭	C♮	F♮	12 ↑ Retrograde Inversion
	1	2	3	4	5	6	7	8	9	10	11	12	

Preliminary exercise

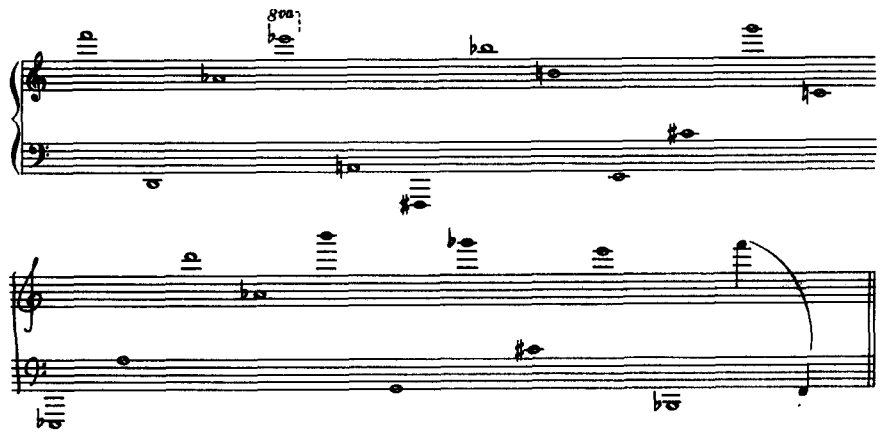


Fig. 36. Piece eight of "Gradus" by S. Adler—table, note of the work 8(a).

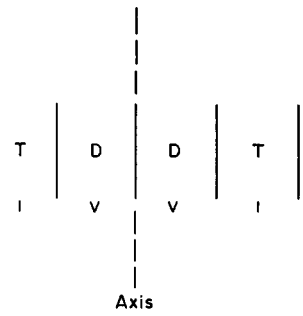


Fig. 37. Symmetry of harmony of tonic-dominant.

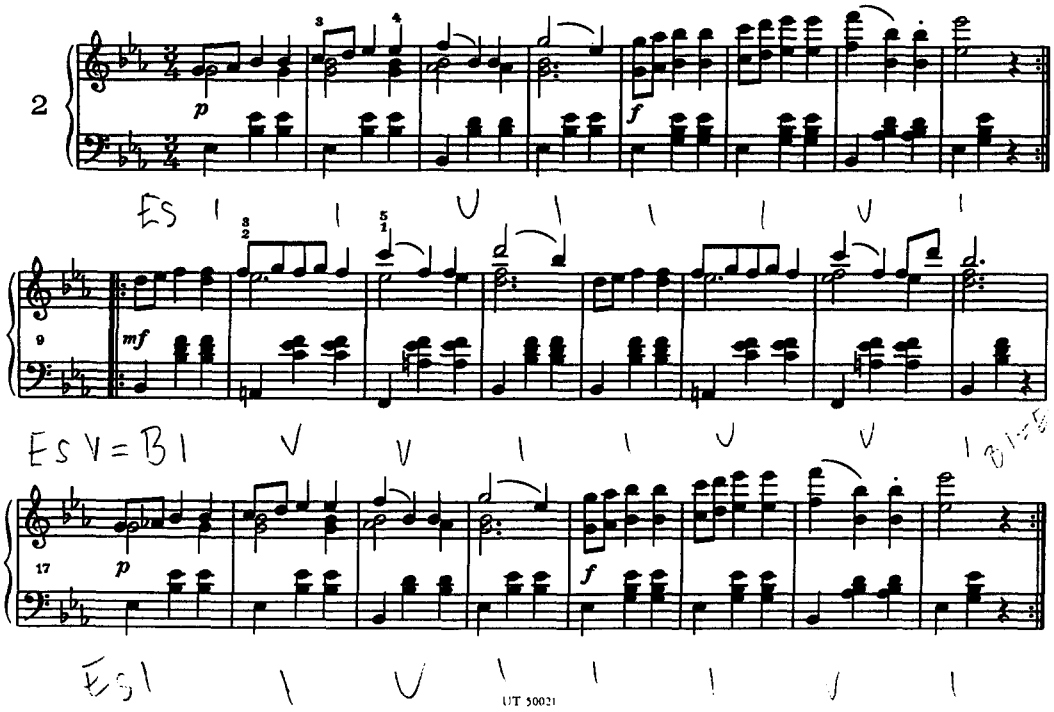


Fig. 38. Note of *Ländler* by Schubert.

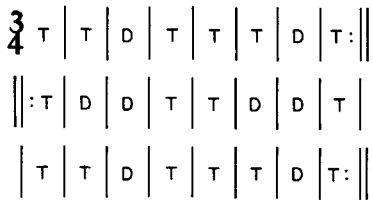


Fig. 39. Harmony scheme of *Ländler* by Schubert.

E flat major	B flat major	E flat major
Basic key	Dominant key	Basic key
8 bars	8 bars	8 bars

Fig. 40. Tonality scheme of *Ländler* by Schubert (example of tonality symmetry).

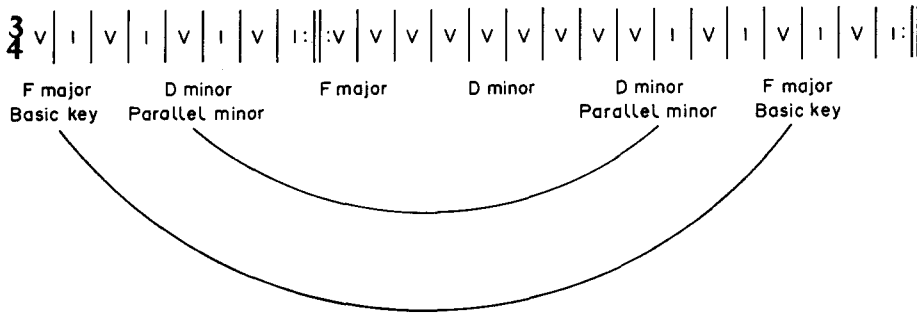


Fig. 41. Harmonies and tonality scheme of *Letzte Walzer* by Schubert.

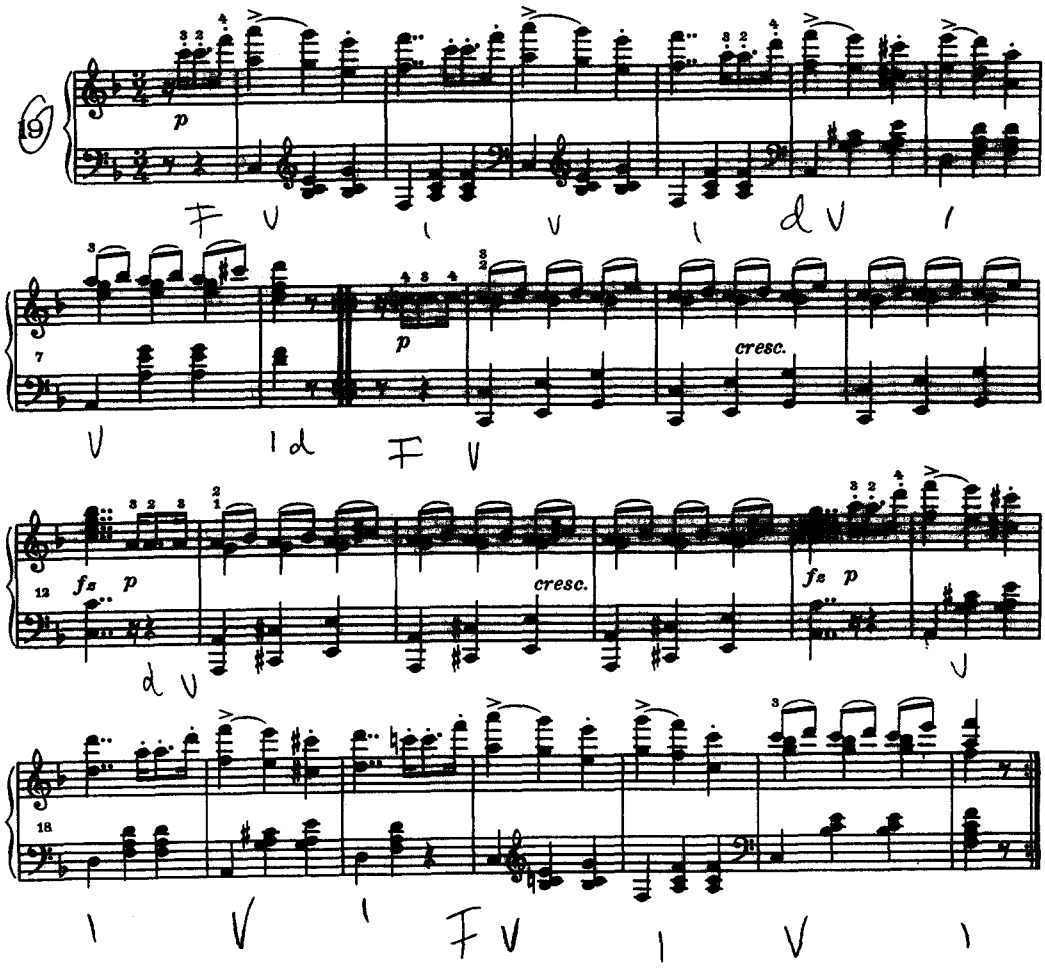


Fig. 42. Note of *Letzte Walzer* by Schubert.

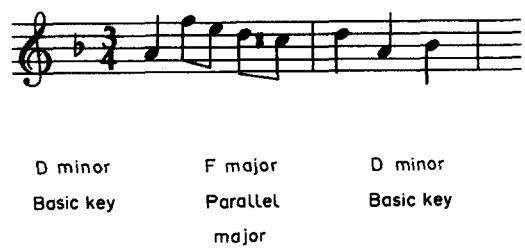


Fig. 43. Bach: Minuet in D minor its note and tonality scheme.

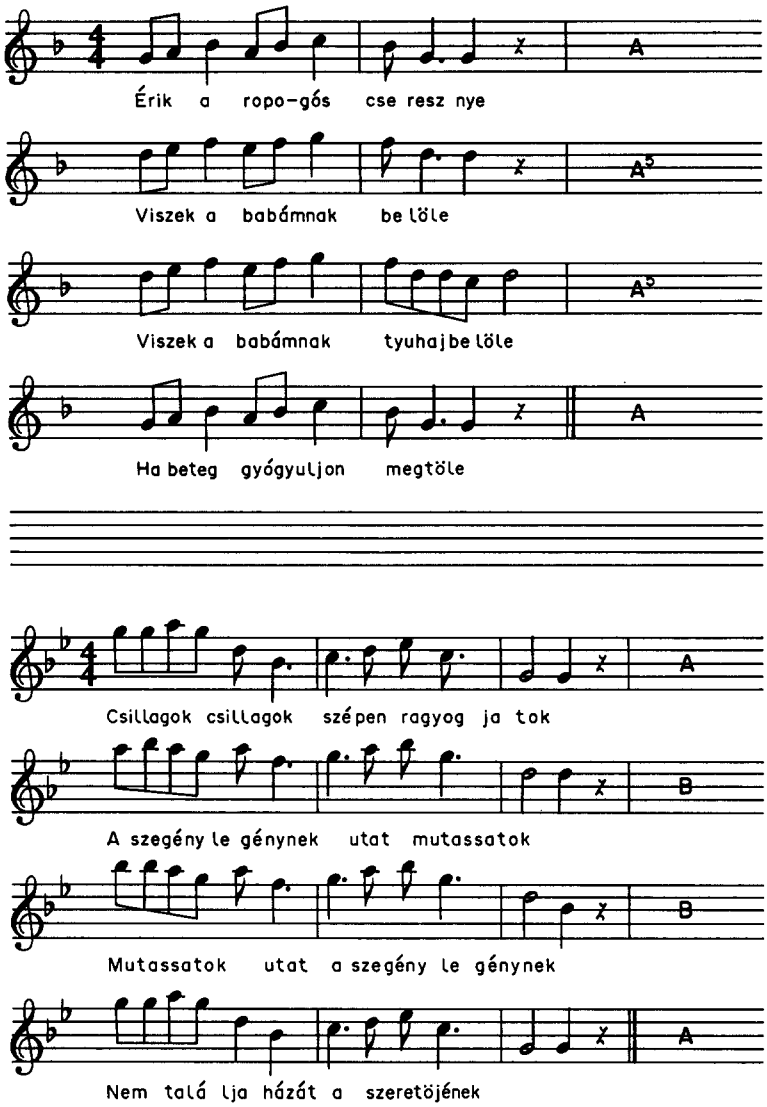


Fig. 44. Notes of two folk-songs.

The piece is characterized by translation symmetry as well, which emphasizes the middle part consisting of fifth (Fig. 42). In the following example we can see minor key which is modulated into its parallel major in the middle part Bach: Minuet (Fig. 43).

All these examples show that symmetries of tonality are in close relationship with the form of the pieces. In each case, however, we have focused on symmetry of tonality only.

SYMMETRY OF FORM

Four-line form

Symmetries are easy to recognize in the four-line Hungarian folk-songs. In particular, two of them are very common: A A^5 A^5 A and A B B A forms.

Translational symmetry is also characteristic of both folk-songs (Fig. 44).

One-unit form

Some pieces consist of a single period, thus creating a one-unit form often divided into a question and an answer. We can take as an example Unisono melody, the first part of *Microcosmos* by

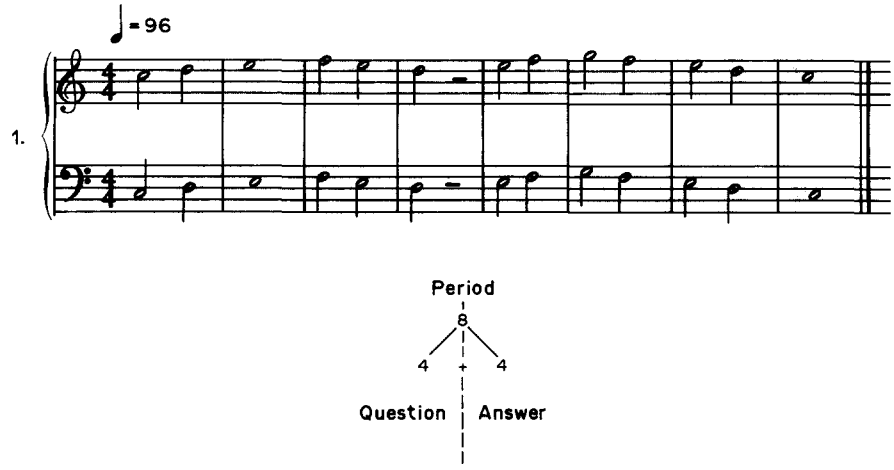


Fig. 45. Note of the piece one of Unisono melody by Bartók and denotion of symmetry of form.

Bartók. The complete piece is a period of eight bars, in which the four-bar question is answered by the other four bars. In this way equilibrium is created. Bartók called it "symmetric balanced structure" (Fig. 45).

Two-unit form

It consists of two mainly repeated periods. From the point of view of symmetry this is the most typical version. It is well illustrated by Mozart's Minuet in C major. Sixteen bars make up the pieces with eight bars in each of the two units. Both units can be divided into 2 + 2 + 4 bars; in this respect the piece is asymmetric. Symmetry and asymmetry generally occur jointly, only their proportion differs. In some cases symmetry is more obvious; some other cases show asymmetry first of all (Fig. 46).

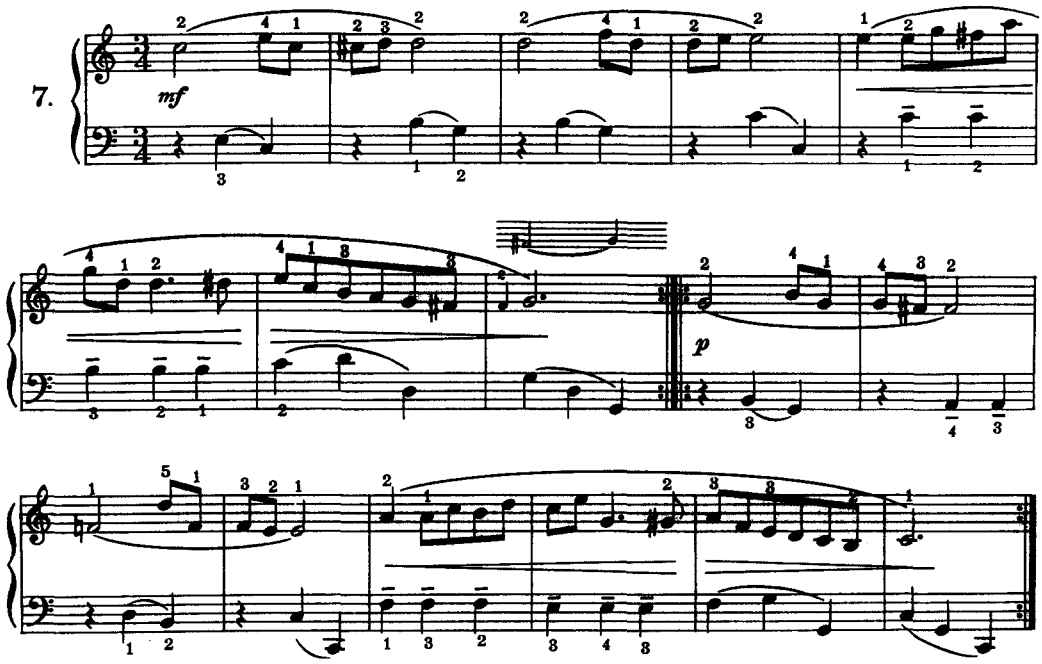


Fig. 46. Note of Minuet C major by Mozart.

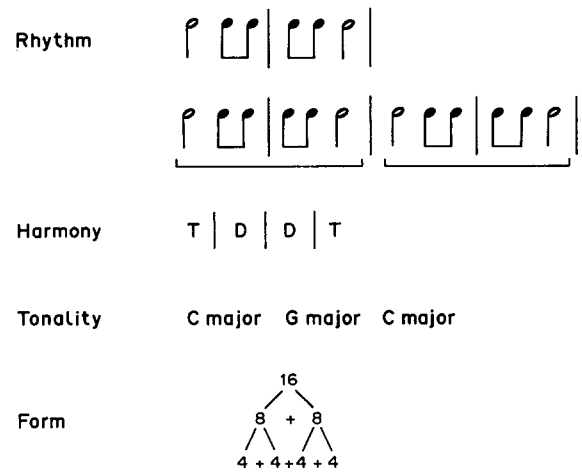


Fig. 47. Table of symmetries in Minuet in C major by Mozart.

Some of the symmetries in this piece relate only to certain details, others relate to the whole piece. Structure of the same principle, both in details and the whole, is demonstrated by the following example (Fig. 47).

Three-unit form

It consists of three sections; the third is either a repeat with no change or a variation. For example, the structure of *Ecossuise* in G major by Beethoven (Fig. 48).

Trio form

It is a multiple, recapitulational structure in which each unit is compound.

Movement *Menuetto al Rovescio* from Sonata No. 41 by Haydn serves as an example (Fig. 49).

The movement is characterized by different types of symmetry. Both in Menuetto and Trio the second period represents retrograde inversion of the first one. Bars corresponding to each other are denoted by numbers. Naturally, the melody is associated by symmetry of harmony, and symmetry characterizes the form as well (Fig. 50).

Rondo form

It consists of theme and interludes with a formula (Fig. 51).

We can mention two couperin-pieces: "*The Reeds*" and "*The Reapers*".

Movements of pieces with more than one movement can be arranged in a symmetrical way. A well-known example is Bach's "*Musikalisches Opfer*", in which the parts are arranged in a symmetry as follows: RICERCAR FIVE CANONS TRIOSONATA TRIOSONATA FIVE CANONS RICERCAR. (Separate study should be devoted to examination and analysis of Bach's

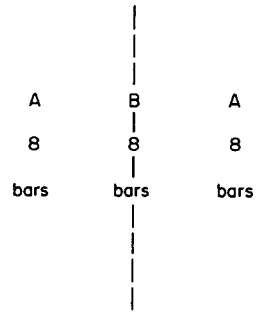


Fig. 48. Symmetry of forms in *Ecossuise* by Beethoven.

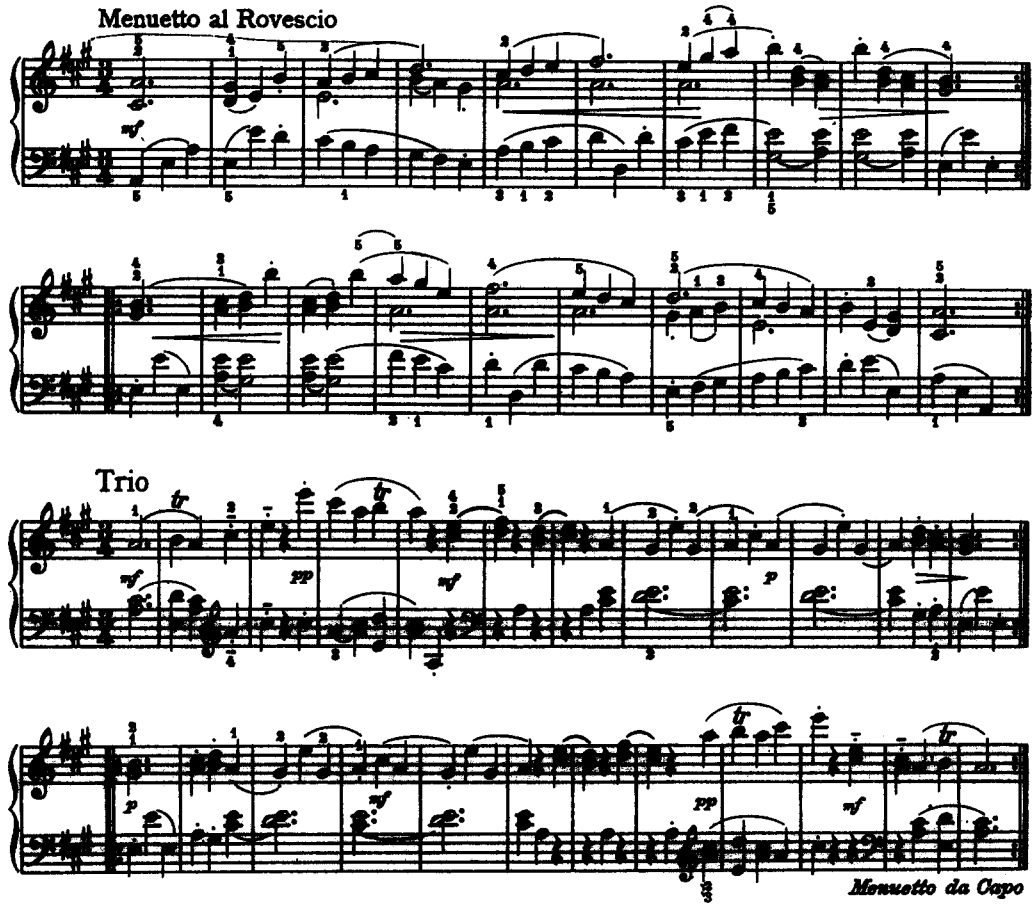


Fig. 49. Note of *Menuetto al Rovescio* by Haydn.

mastery to structure canons and fugues from the point of view of symmetry. Great variety of different symmetries can be found in his works and acme can be observed in the cycle "*Die Kunst der Fuge*".)

We can take an example of multimovement symmetry from the twentieth century. It is *Canticum Sacrum* by Stravinsky, which was first performed in St Marks cathedral in Venice. It has five movements, and some of the criticians found its structure the same as that of the cathedral having five domes, since the last movement of the work is nearly an exact retrograde-version of the first one. So its scheme is like the one shown in Fig. 52.

GOLDEN SECTION

When we speak of symmetry we must make mention of the golden section. It is one of the most typical ratios according to which the whole is to the major part as the major part to the minor one. This ratio in itself is asymmetric, however if positive and negative golden cuts are present together—being equalized—they make up a symmetry (Fig. 53).

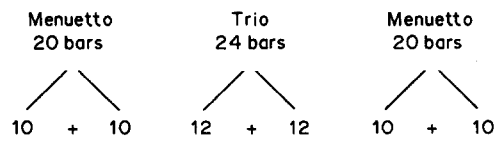


Fig. 50. Figure of symmetry of form of the Haydn Minuet.

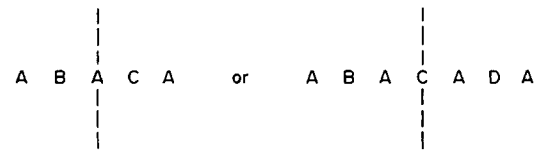


Fig. 51. Rondo form.

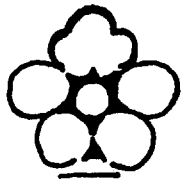


Fig. 52. Scheme of a cyclic piece. (Taken from the book *Stravinsky* by White.)

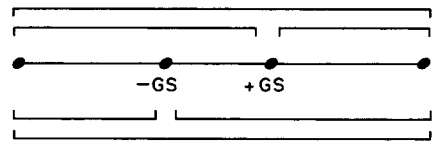


Fig. 53. Golden section—figure of explanation.

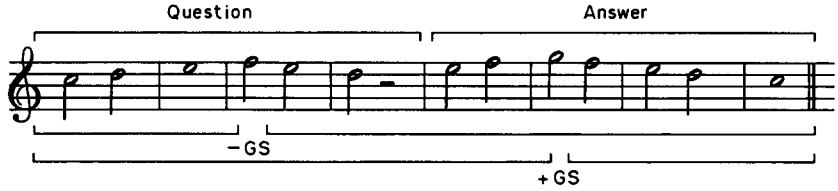


Fig. 54. Golden cuts of piece one of *Microcosmos* by Bartók.

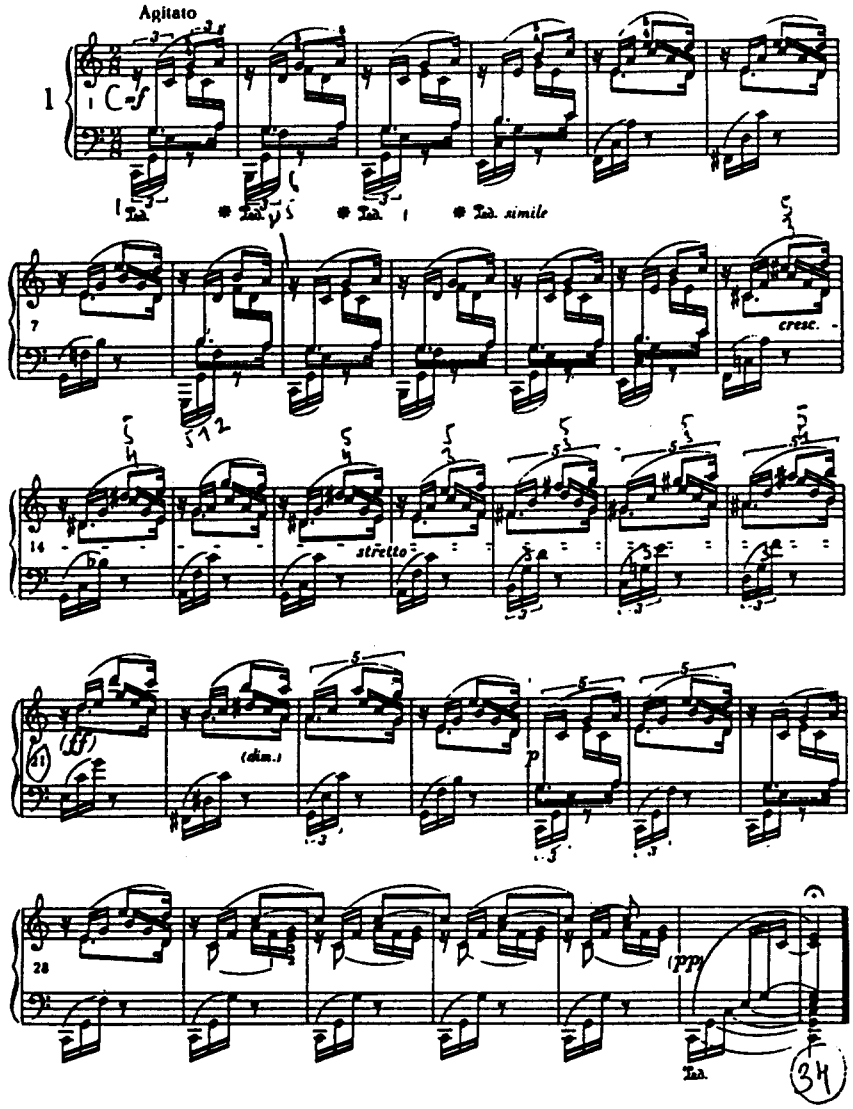


Fig. 55. Note of Prelude 1 by Chopin.

In piece one of *Microcosmos* by Bartók the peak tones represent the intersections of golden cut, which can be found in bar three and after bar five, respectively (Fig. 54).

The peak tone in question is the F″ which is the intersection of negative golden cut, while that of the answer is the G″ which is the intersection of positive golden cut throughout the piece.

Another symmetry in golden section is that regularity is constantly repeated. For example in the case of Fibonacci-numbers: 1, 2, 3, 5, 8, 13, 21, 34, 55 . . . any new number in the sequence is made up of the preceeding two numbers: $1 + 2 = 3$; $2 + 3 = 5$; $3 + 5 = 8$ etc.

The next piece is Prelude 1 by Chopin. Here the golden section can be detected in dynamics and melody, based upon the number of bars. It has 34 bars starting with Mezzoforte; this continuously increases up to bar 21. Dynamics fortissimo in bar 21 represents the peak—here we find the highest note as well. From this point on both melody and dynamics are decreased (Figs 55 and 56).

The positive golden cut in this case represents an asymmetric ratio formally, as it stands alone. At the same time a translational symmetry can be found throughout the Prelude.

Another example of simultaneous presence of symmetry and asymmetry is taken from *Dance of Kites* in Vol. III of *Microcosmos* by Bartók (Fig. 57).

In all practicality it can be divided into three parts each consisting of eight bars. In the first eight bars four tones are reflected across a horizontal axis between two G tones (Fig. 58).

In the second unit the two constantly sounding tones are E′ and g sharp with five tones reflected (Fig. 59).

In bars 14–15 we can find the positive golden cut preparing for part three. Here, for a short time, a reflection across the vertical axis can be observed. Then in the last four bars reflection across the horizontal axis returns.

In the third unit the organ point is represented first by G″ and b, then by E″ and D″. At the end the two outside G tones (Fig. 60).

The relationship of the parts consisting of constantly sounding (pedal point) and changing tones in the three units (Fig. 61).

The starting position of the first unit reappears in the second one, however, in a contrary way. While the third unit unifies both versions on the pattern of thesis–antithesis–synthesis.

However, this unification is created not only by a simple repetition of the former two units but by means of elaboration as consummation, therefore ending in perfection of the musical material.

Supposingly, this rich variety of musical symmetries is not by chance, since all the tempered tonal system is built up symmetrically.

Lajos Bardos discussed this issue in details in one of his essays published in the periodal *Parlando* (February, 1979). (Figures are taken from his essay.)

White keys of a tempered piano make up the diatonic heptatonic scale, which can be presented as a perfect fifth as well:

F C G D A E H

Black keys make up the pentatonality:

F sharp/G flat C sharp/D flat G sharp/A flat
D sharp/E flat A sharp/B flat.

The two scales symbolized in circle of fifths make up the tempered dodecaphony (Fig. 62).

Tones D and G sharp, opposite each other, denote two centres of symmetry.

Tone D is centre of symmetry of the diatonic scale (Fig. 63).

Tone G sharp (A flat) is the centre of symmetry of the pentatonic scale (Fig. 64).

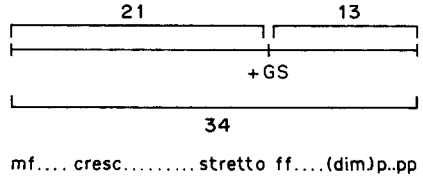


Fig. 56. Figure of explanation of Prelude 1 by Chopin.

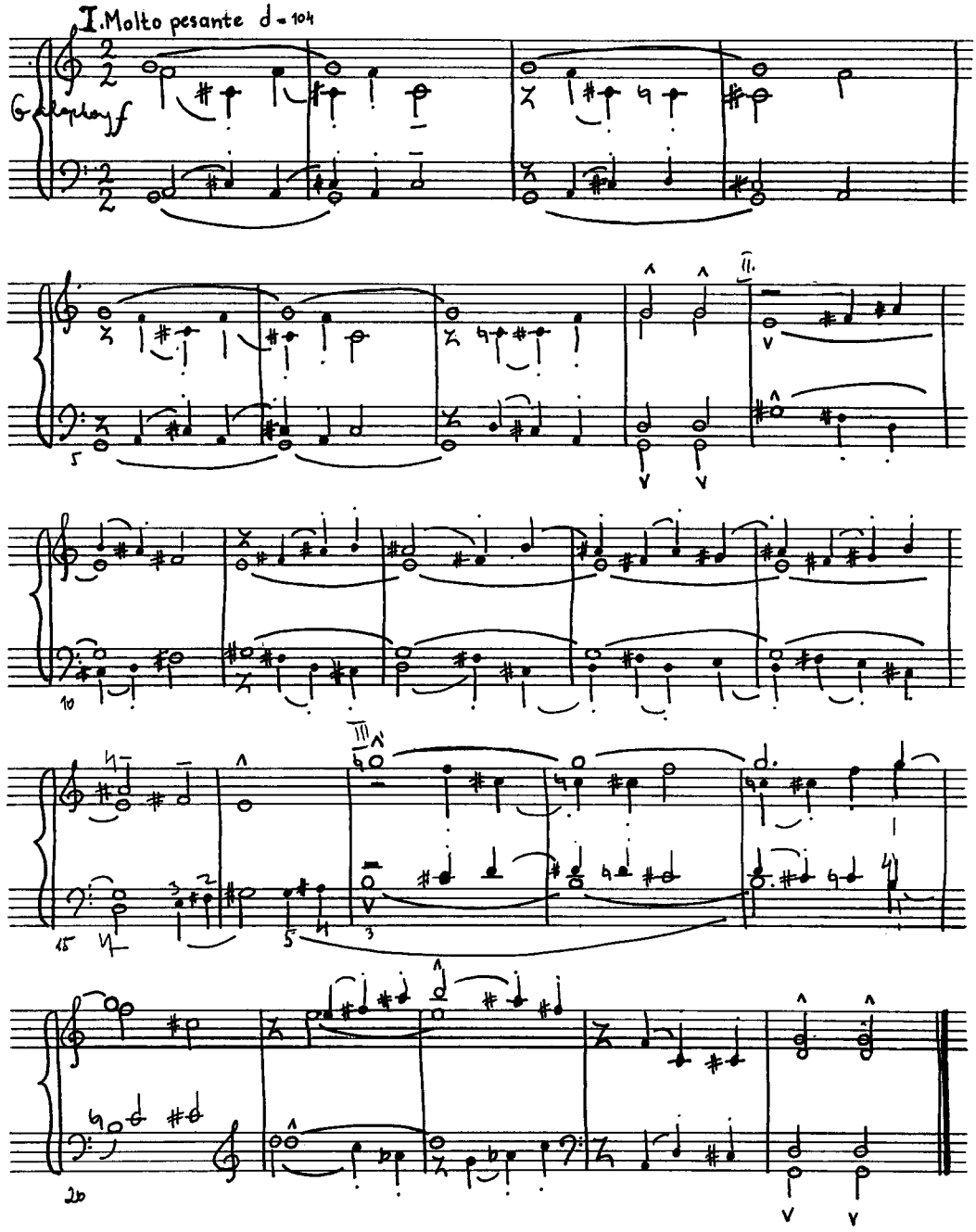


Fig. 57. Bartók: *Dance of Kites*—note.

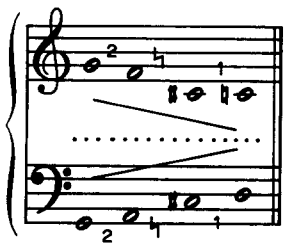


Fig. 58. Analysing note of the first part of *Dance of Kites* by Bartók.

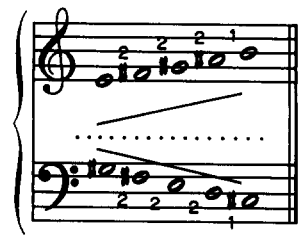


Fig. 59. Analysing note of the second part of *Dance of Kites* by Bartók.

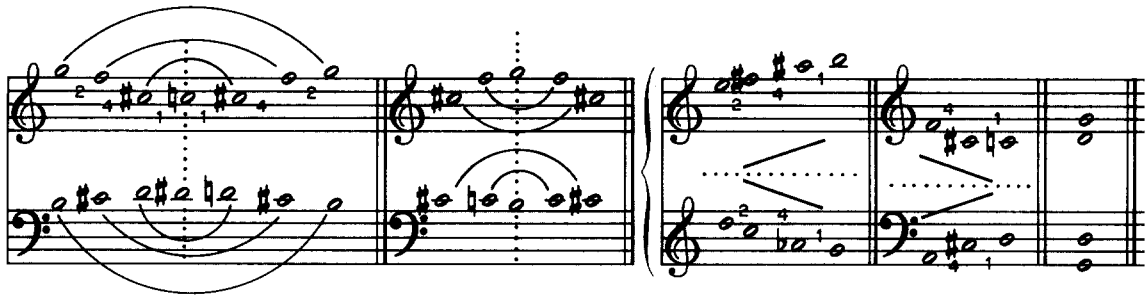


Fig. 60. Analysing note of the third part of *Dance of Kites* by Bartók.

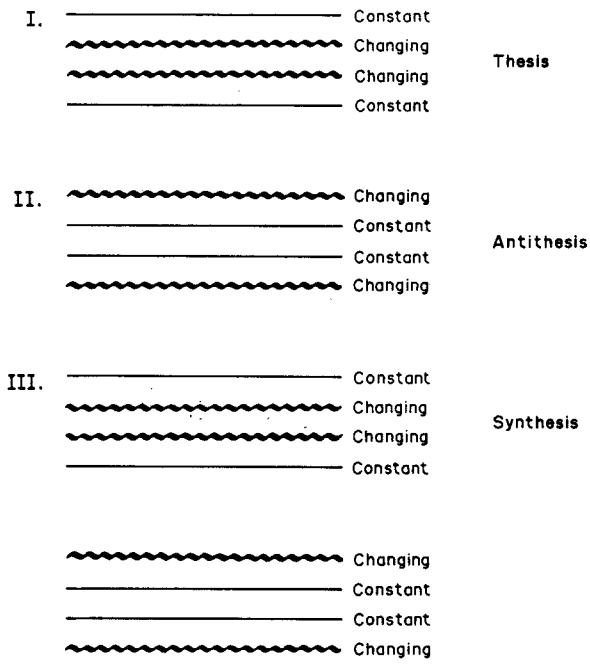


Fig. 61. Figure of explanation of *Dance of Kites* by Bartók.

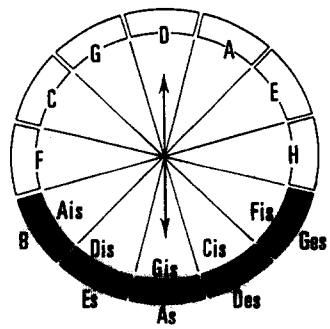


Fig. 62. Illustration in circle of fifths.

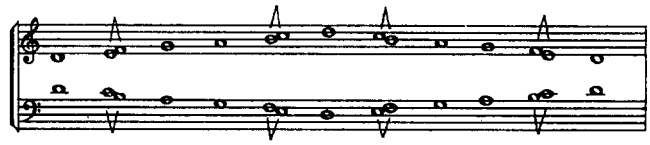


Fig. 63. Symmetry of diatonic scale—note.

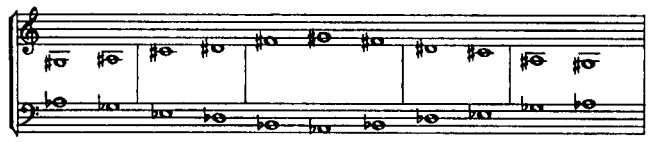


Fig. 64. Symmetry of pentatonic scale—note.

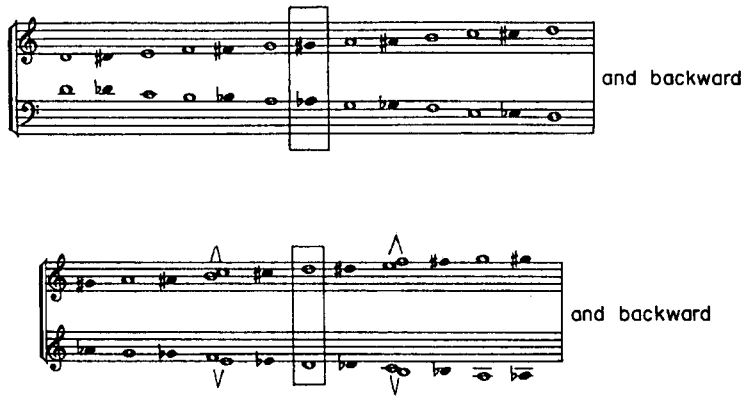


Fig. 65. Symmetry of chromatic scale—two notes.

From these, we may conclude that the chromatic scale is also symmetric in case of D and G sharp (Fig. 65).

The symmetry centre of the white keys is the tone G sharp, while D is that of the black keys.

The essay we are referring to does not deal with different types of symmetries of time and space. However, it takes conditions of time and space of music occurrences preconditioning each other. Within it, time symmetry is generally characterized by succession and reflection through a vertical axis; while simultaneity and reflection through a horizontal axis characterize space symmetry.

We could use numerous other examples of symmetries in music but the types presented above are the simplest occurrences.

Its importance is indicated by the fact that from the very first moment of education and training it occurs and reoccurs time by time as a basic problem in different ways, and examples can be found in pieces by each composer.

However it is very important that pupils, in the process of teaching, should understand and recognize the different structures including those other than symmetry. With this aim in mind, e.g. when learning types of symmetry it is very reasonable to take the possibilities enumerated in the study as a starting point and exercise them in improvisations and composing. The experience is that such kind of work and activity deeply influence pupils from emotional point of view.

Thus improvising, composing and interpreting jointly promote the progress in learning the tongue and bulk of music.